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Orthogonality between Scales and Wavelets in a Representation for Correlation Functions. The Lattice Dipole Gas and $(\nabla\phi)^4$ Models

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Exact formulas for the correlation functions of lattice scalar field models in $\mathbf{Z}^d$, $d \geqslant 3$, such as the dipole gas and anharmonic crystal are derived in terms of the effective action generated after n applications of the block renormalization group transformation. Utilizing the orthogonality between different momentum scales (relations due to the wavelets implicit in the structure of the block renormalization group transformation), the formulas are quite simple, isolate the dominant term, and, in the thermodynamic and $n \to \infty$ limits, reduce the analysis to local estimates of the effective action. Based on a large-small field analysis, the two-point function is determined and it is shown how to extend the results to general correlations. The results proved here show the usefulness of the "orthogonality-of-scales" property for the study of correlation functions.

KEY WORDS: Orthogonality between scales; correlation functions; block renormalization group; dipole gas.

1. INTRODUCTION

The present paper establishes rigorously the thermodynamic limit of a useful representation for the correlation functions of some lattice scalar field theories such as the dipole gas and $(\nabla\phi)^4$ models, $d \geqslant 3$. The analysis here completes the perturbative study described in a previous article,[(1)] and the treatment is carried out in the framework of the large-small field analysis of refs. 2 and 3. Only the two-point function is studied in detail, but we indicate the extension to general correlations.

Our aim is to show the usefulness of the orthogonality between different momentum scales for the study of correlation functions (property

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due to the lattice wavelet structure implicit in the block renormalization group). We expect later to obtain similar representations for more complicated models and other renormalization groups.

Let us summarize the results. The final representation leads to exact formulas for the correlation functions, not only to asymptotic approximations. Different momentum scales represented by certain operators in these formulas are shown to be related to mutually commuting orthogonal projections which are associated to lattice wavelets. This "natural" orthogonality between scales makes the formulas quite simple, and their analysis poses no difficulty. The terms giving the correct long-distance behavior and the subdominant contributions are separated in the correlation functions, which are determined only by the limit of field derivatives of the effective action at zero field and by a sequence of wavefunction renormalization constants. The long-distance behavior for the correlations confirms existing results,[4,5] but the error term is smaller (i.e., we have a better control of the correction to the terms associated with the massless Gaussian theory).

The models to be considered are given by the finite lattice scalar field theory expressed by the Hamiltonian

$$\mathscr{H}(\phi) = \tfrac{1}{2} b_0(\phi, \Delta\phi) + V(\phi) \tag{1.1}$$

where $\phi(x) \in \mathbf{R}$; $x \in \Lambda_N$, $\Lambda_N \equiv [-L^N/2, L^N/2]^d \cap \mathbf{Z}^d$, L odd, $d \geqslant 3$, and $\Delta \equiv \partial^\dagger \partial$ (considering Dirichlet boundary conditions; for periodic boundary conditions we shall include the zero mode regulator; see ref. 2). V is assumed to be a functional of the vector field $\partial_\mu \phi(x)$, $x \in \Lambda_N$, invariant under lattice translations, rotations, and reflections, even, and vanishing at zero,

$$V(\phi) = \sum_{\mu; x \in \Lambda_N} v(\partial_\mu \phi(x)) \tag{1.2}$$

The lattice dipole gas and anharmonic crystal are examples of such interactions.

To obtain the representation, we follow the flow of the generating function

$$\mathrm{Z}(h) = \int \exp[-H(\phi)]\, D\phi \tag{1.3}$$

$H(\phi) = \mathscr{H}(\phi) - (h, \phi)$, $D\phi = \prod_{x \in \Lambda_N} d\phi(x)$, by the block spin renormalization transformations given by, in the first step,

$$\exp[-H^1(\psi)] = \frac{\int \exp[-H(\phi)]\, \delta(C\phi - \psi)\, D\phi}{(\text{numerator with } \psi \equiv h \equiv 0)} \tag{1.4}$$

where $\delta(C\phi-\psi)=\prod_{x\in\Lambda_{N-1}}\delta(C\phi(x)-\psi(x))$, and $C\phi(x)$ is the rescaled average in the block b^L_{Lx} of side L, centered at $Lx\in\Lambda_N$,

$$C\phi(x)=L^{(d-2)/2}L^{-d}\sum_{y\in b^L_{Lx}}\phi(y) \tag{1.5}$$

$L^{(d-2)/2}$ is the scaling factor (using the canonical field dimension).

After n steps ($n\leqslant N$), making explicit at each step the minimum of the effective action [calculated after discarding the small perturbative potential and considering the constraint $\delta(C\phi^j-\phi^{j+1})$, where ϕ^{j+1} is the block field at the $(j+1)$th scale], and separating the marginal terms (local quadratic part), we obtain (see ref. 1 for details)

$$\mathsf{Z}(h)=c\exp[\tfrac{1}{2}(h,P_nh)]$$
$$\times\int\exp\{-V^n(\partial_\mu[M_n\phi+G_nh])-\tfrac{1}{2}b_n(\phi,\Delta_n\phi)\}\,D\phi \tag{1.6}$$

where c does not depend on h; b_n is the wavefunction renormalization constant at the nth step; V^n is the nth perturbative potential minus its local quadratic part; P_n and G_n are given by

$$P_n=b_n^{-1}\Delta^{-1}-\sum_{j=0}^{n-1}\frac{1}{b_n}\left(\frac{b_n-b_j}{b_j}\right)^2 M_j\Gamma_jM_j^\dagger \tag{1.7}$$

$$G_n=b_n^{-1}\Delta^{-1}-\sum_{j=0}^{n-1}\frac{1}{b_n}\left(\frac{b_n-b_j}{b_j}\right)M_j\Gamma_jM_j^\dagger \tag{1.8}$$

with

$$\Gamma_j=\Delta_j^{-1}-\Delta_j^{-1}C^\dagger\Delta_{j+1}C\Delta_j^{-1} \tag{1.9}$$

$$\Delta_j=(C_j\Delta^{-1}C_j^\dagger)^{-1} \tag{1.10}$$

$$M_j=\Delta^{-1}C_j^\dagger\Delta_j \tag{1.11}$$

where C_j is the rescaled average over a block of side L^j, and is given by (1.5) replacing L by L^j. We remark that the limit of Δ_n as n goes to infinity is the fixed point associated with the continuum Laplacian, and that P_n contains the main terms for the correlation functions (as we hope, and prove later, since it is formed by the minimum of the effective action separated at each step). It is shown in refs. 1 and 2 that

$$|M_j\Gamma_jM_j^\dagger(x,y)|\leqslant L^{-j(d-2)}\exp(-\alpha' L^{-j}|x-y|),\qquad \alpha'>0$$

so that, roughly speaking, the jth term of (1.7) and (1.8) gives the contribution around the momentum scale L^{-j}.

We shall emphasize that the operators $M_j \Gamma_j M_j^\dagger$ are associated with the Laplacian decomposition

$$\Delta^{-1} = \sum_{j=0}^{n-1} M_j \Gamma_j M_j^\dagger + M_n \Delta_n^{-1} M_n^\dagger \tag{1.12}$$

with

$$|M_n \Delta_n^{-1} M_n^\dagger(x, y)| \leqslant c L^{-n(d-2)} \tag{1.13}$$

(see Section 3), and that $\mathscr{P}_j \equiv \Delta^{1/2}(M_j \Gamma_j M_j^\dagger)\, \Delta^{1/2}$ and $\mathscr{Q}_n \equiv \Delta^{1/2}(M_n \Delta_n^{-1} M_n^\dagger)\, \Delta^{1/2}$ are mutually orthogonal projections

$$\mathscr{P}_j \mathscr{P}_i = \delta_{ij} \mathscr{P}_j, \quad \mathscr{P}_j^\dagger = \mathscr{P}_j, \quad \mathscr{Q}_n^2 = \mathscr{Q}_n, \quad \mathscr{Q}_n^\dagger = \mathscr{Q}_n, \quad \mathscr{Q}_n \mathscr{P}_j = \mathscr{P}_j \mathscr{Q}_n = 0 \tag{1.14}$$

(for $j = 0,..., n-1$). The lattice wavelets (which we have claimed to be implicit in the structures presented here) are given by $f_j \equiv \Delta^{1/2} M_j u$ (for any u in Λ_{N-j}, with $Cu = 0$), the eigenfunctions of $\mathscr{P}_j$ above, and by $h_n \equiv \Delta^{1/2} M_n v$ (for any v in Λ_{N-n}), the eigenfunctions of $\mathscr{Q}_n$ (i.e., the wavelets are the spectral decomposition of these operators). Note that the orthogonality relation between scales [which is needed to obtain the representation (1.6)] can be stated as an orthogonality of wavelets on different scales. However, we find it notationally more economical and conceptually clearer to use the orthogonality as given in (1.14). Translation properties of the eigenfunctions and commentaries about the formulas above are presented in ref. 1; see ref. 6 for more details about the relation between wavelets and the block renormalization group.

One must also emphasize the simplicity of the generating function (1.6). Due to the orthogonal relations between operators associated with different momentum scales (in other words, to the connection with wavelets), the final formula is quite simple: there are two propagators [given by the formulas (1.7) and (1.8)] and there is no mixing between scales.

Differentiating $\ln \mathsf{Z}(h)$, we generate the k-point truncated function

$$\begin{aligned}\left\langle \prod_{i=1}^{k} \phi(x_i) \right\rangle^T &\equiv \partial^k \ln \mathsf{Z}(h)/\partial h(x_1) \cdots \partial h(x_k)|_{h\equiv 0} \\ &= \delta_{k,2} P_n(x_1, x_2) - D^k_{y_1 \cdots y_k} V^n(0) \prod_{i=1}^{k} \partial_{\mu_i} G_n(y_i, x_i) + R_{kNn}\end{aligned} \tag{1.15}$$

∂_{μ_i} acting on y_i, and

$$D^k_{y_1 \cdots y_k} V^n \equiv \partial^k V^n / \partial\chi(y_1) \cdots \partial\chi(y_k), \qquad \chi \equiv \partial_\mu M_n \phi$$

$$\begin{aligned} R_{kNn} = {} & \frac{\partial^k}{\partial h(x_1) \cdots \partial h(x_k)}\bigg|_{h=0} \\ & \times \ln \int \exp\left\{ -\sum_{p=1}^{\infty} \frac{1}{p!} \left[D^p_{y_1 \cdots y_p} V^n(\partial_\mu M_n \phi) - D^p_{y_1 \cdots y_p} V^n(0) \right] \right. \\ & \left. \times \prod_{i=1}^{p} \partial_{\mu_i} G_n(y_i, x_i)\, h(x_i) \right\} \exp\left[-V^n(\partial_\mu M_n \phi) - \frac{b_n}{2}(\phi, \Delta_n \phi) \right] D\phi, \end{aligned} \tag{1.16}$$

where we made explicit the derivatives of the effective potential at zero field.

In the present article we intend to prove that $\lim_{N,n \to \infty} R_{kNn} = 0$, besides the well-known fact that the canonical scaling limit is given by the massless Gaussian Euclidean field. Intuitively, the vanishing of R_{kNn} is expected since it represents, roughly, contributions from the momentum scale $[0, L^{-n}]$.

The rest of the paper is organized as follows: in Section 2 we give the Gibbs factor representation that is carried over by the renormalization transformations, and state inductive hypotheses which allow us to follow the flow of the derivatives of effective potential. In Section 3, using these hypotheses, we establish in detail the thermodynamic limit for the two-point function and carefully study its asymptotic behavior. Section 4 is devoted to technical proofs of the inductive hypothesis, and Section 5 to general correlations and final comments.

2. SMALL-LARGE FIELD ANALYSIS AND INDUCTIVE HYPOTHESIS FOR THE DERIVATIVES OF THE EFFECTIVE POTENTIAL

We extend here the small-large field analysis of refs. 2 and 3 for the flow of the derivatives of the effective potential in order to obtain a good control of these terms (i.e., of $D^k_{y_1 \cdots y_k} V^n$, for arbitrary n).

The considered method is a nonperturbative treatment of the block renormalization group, and was developed to deal simultaneously with the nonlocal potentials (created by the renormalization transformation) and with the positivity of the effective actions. Cluster expansions and analyticity properties of the effective interactions are used to solve these problems, but after separating the regions where the large block fields can spoil the convergence of the expansions (these regions are controlled by iterative bounds).

Let us introduce now the representation for the Gibbs factor (related to the small-large field division) which is carried over by the renormalization group transformations. The notation is similar to that in ref. 2 and we use the structures and results proved there. Thus, reading ref. 2 is necessary to understanding the present article (specifically, Section 4).

We shall consider subsets D, X, Y,... of lattices $L^{-n}\Lambda_N$ or Λ_{N-n} being a union of blocks Δ of side L^{N_0}, centered at points $L^{N_0}\Lambda_{N-n-N_0}$ (unless otherwise stated). By $|X|$ we understand the number of blocks Δ in X, and by $\mathscr{L}(X)$ the length of the shortest connected graph on the centers of blocks Δ in X. By $\bar{B}$ we denote the union of blocks Δ intersecting B (B a subset of $L^{-n}\Lambda_N$ or Λ_{N-n}). Note that now we are considering lattices such as $L^{-n}\mathbf{Z}^d$, not only unitary lattices.

We take complex vector fields χ^n on $L^{-n}\Lambda_N$ (although only real fields interest us, bounds are easily obtained in a complex manifold), and for $X\subset L^{-n}\Lambda_N$ define the small-field region as

$$\begin{aligned}\mathscr{K}_n(X)=\{\chi^n\equiv\chi^n_\mu(z),\, z\in X\colon &|\chi^n_\mu(z)|<(n_0+n)^\nu\\ &|\nabla_\zeta\chi^n_\mu(z)|<c_1(n_0+n)^{\nu+d}\qquad\text{if}\quad z+L^{-n}e_\zeta\in X\}\end{aligned}\tag{2.1}$$

where c_1 is a constant properly chosen, and $\nu>\frac{1}{2}d^2+$ is fixed.

Given $\nabla\psi^n$, where

$$\psi^n(z)=(\mathscr{A}_n\phi^n)(z)\tag{2.2}$$

ϕ^n is the real block field at the nth step, and $\mathscr{A}_n$ the minimizer on $L^{-n}\Lambda_N$,

$$\mathscr{A}_n(z,\, y_l)=L^{n(d-2)/2}M_n(L^nz,\, y_l)\tag{2.3}$$

($z\in L^{-n}\mathbf{Z}^d$ and $y_l\in\mathbf{Z}^d$); we define the large-field region $D_n(\nabla\psi^n)$ as the smallest subset of $L^{-n}\Lambda_N$ such that

$$|\nabla_\mu\psi^n(z')|\leqslant(n_0+n)^\nu\exp[\alpha' d(z,\, z')]\tag{2.4}$$

for each $z\notin D_n(\nabla\psi^n)$, where $\alpha'=\varepsilon\beta'$, for ε a small constant depending only on the dimension, and β' an L-dependent constant associated with the exponential decay of various kernels (proved in ref. 2), such as

$$\mathscr{A}_n(z,\, y_l)|\leqslant c\exp[-\beta' d(z,\, y_l)]\tag{2.5}$$

We shall separate the nonlocal quadratic term from the potential defining (remember that we have already separated the local quadratic part)

$$\tilde{V}^n(\chi^n)\equiv V^n(\chi^n)-\tfrac{1}{2}(\nabla\chi^n,\, K_n\nabla\chi^n)\tag{2.6}$$

Given $\nabla\psi^n$, a set $D \supset D_n(\nabla\psi^n)$, and $\tilde{\chi}^n \in \mathscr{K}_n$, we write the representation for the Gibbs factor

$$\exp[-\tilde{V}^n(\nabla\psi^n + \tilde{\chi}^n)] = \sum_{\{X_j\}} \prod_j g^{nD}_{X_j}(\nabla\psi^n + \tilde{\chi}^n) \times \exp\left[-\sum_{Y \subset \sim \bigcup_j X_j} \tilde{V}^n_Y(\nabla\psi^n + \tilde{\chi}^n)\right] \tag{2.7}$$

where $\sum_{\{X_j\}}$ runs over the sets of disjoint $X_j \subset L^{-n}\Lambda_N$; $D \cap X_j$ is a union of connected components of D; $D \subset \bigcup_j X_j$, $g^{nD}_{X_j}$ depends on χ^n through its restriction to X_i (the same for $\tilde{V}^n_Y$ and Y).

Now we state the inductive hypotheses for the effective potential (i.e., for the Gibbs factor representation) which allow us to prove the thermodynamic limit of the correlation formulas. In order to avoid unnecessary notational complications, we shall exhibit a detailed study for the two-point function only (but in Section 5 we extend the arguments and the hypotheses to a general correlation).

The statement involves several parameters: δ, L, N_0, r, B, E, n_0. For while we assume that $0 < \delta < 1$, $L > L_0(\delta)$, $N_0 > \tilde{N}_0(\delta, L)$, $r > r_0(\delta, L, N_0)$, $B < B_0(L, N_0)$, $E > E_0(L_0, N_0, r)$, and $n_0 > \tilde{n}_0(\delta, L, N_0, B, r, E)$, where L_0, $\tilde{N}_0$, r_0, B_0, E_0, and $\tilde{n}_0$ do not depend on the volume L^{Nd}, nor on n (the renormalization step). Later we take δ as a function of L (precisely, there is a constant c so that $\delta^n \leqslant cL^{-2n}$ for all n). All constants are denoted by c, and none depends on the volume (N).

We make four inductive assumptions:

1$_n$. g^{nD}_X is an even analytic functional on $\mathscr{B}_n(D, X, 1)$, where

$$\mathscr{B}_n(D, X, a) = \bigcup_{D_n(\nabla\psi^n) \subset D} (\nabla\psi^n|_X + a\mathscr{K}_n(X)) \tag{2.8}$$

For all D_1 with $D_1 \cap D = \bigcup_j D \cap X_j$, X_j disjoint, and $\chi^n = \nabla\psi^n + \tilde{\chi}^n$, $D_n(\nabla\psi^n) \subset D$, $\tilde{\chi}^n|_{X_j} \in \mathscr{K}_n(X_j)$, we have

$$\prod_j |g^{nD}_{X_j}(\chi^n)| \leqslant \exp\left[k_n \mathscr{D}_n(D_1, \nabla\psi^n) - 2\alpha' \sum_j \mathscr{L}(X_j) + E \sum_j |D \cap X_j|\right] \tag{2.9}$$

where

$$\mathscr{D}_n(D, \chi^n) = \left(\int_D dz + \int_{\partial D} d\sigma(z)\right) \sum_\mu (\chi^n_\mu(z))^2 \tag{2.10}$$

$$\int_{\partial D} d\sigma(z) \equiv \sum_{z,\mu} L^{-n(d-1)} \qquad \begin{array}{l} \text{for } z \in D, z+L^{-n}e_\mu \notin D \\ \text{or } z \notin D, z+L^{-n}e_\mu \in D \end{array} \tag{2.11}$$

$$k_{n+1} = k_n + c(n_0+n)^{-2} \tag{2.12}$$

and α' as given in (2.4).

2$_n$. $\tilde{V}^n_Y$ is an even analytic functional on $2\mathscr{K}_n(Y)$, bounded by

$$|\tilde{V}^n_Y| \leqslant \delta^{n_0+n} \exp[-2\alpha' \mathscr{L}(Y)] \tag{2.13}$$

where

$$\tilde{V}^n_Y(0) = \frac{\delta^2 \tilde{V}^n_Y}{\delta\chi^n_\mu(z)\, \delta\chi^n_\nu(z')}(0) = 0 \tag{2.14}$$

3$_n$. The irrelevant quadratic term satisfies

$$|K_n(z, z')| \leqslant (n+1)^{-1} \delta^{n_0+n} \sum_{k=0}^{n} \exp[-2\alpha' L^k d(z, z')]\, L^{dk} \tag{2.15}$$

and for the wavefunction renormalization

$$|b_n - b_{n+1}| \leqslant \delta^{n_0+n} \tag{2.16}$$

4$_n$. For $\tilde{V}^n_Y$ on $2K_n(Y)$, z, $z' \in Y$,

$$\left| \frac{\partial \tilde{V}^n_Y}{\partial \chi^n_u(z)}(\chi^n) \right| \leqslant c(n_0+n)^{4v+4d} L^{-n(d+1)} \delta^{n_0+n} \exp[-2\alpha' \mathscr{L}(Y)] \tag{2.17}$$

$$\left| \frac{\partial^2 \tilde{V}^n_Y}{\partial \chi^n_\mu(z)\, \partial \chi^n_\zeta(z')}(\chi^n) \right| \leqslant c(n_0+n)^{4v+4d} L^{-n(d+2)} \delta^{n_0+n} \exp[-2\alpha' \mathscr{L}(Y)] \tag{2.18}$$

(for z not in Y the derivative vanishes).

The hypotheses **1**$_n$–**3**$_n$ are considered and proved in ref. 2 (for each n so that $0 < n < N - N_0$). Note that **4**$_n$, i.e., an extra contractive factor associated with the potential derivatives, is intuitively expected: naively taking, e.g., the potential $\lambda_n \int [\chi^n(z)]^k\, dz$ at the nth step with the derivative $\partial/\partial\chi^n(u)$, we pick up a factor L^{-nd} obtaining $k\lambda_n L^{-nd}[\chi^n(u)]^{k-1}$.

3. THE TWO-POINT FUNCTION

Before **4**$_n$, let us show that, in the thermodynamic limit, the two-point correlation is determined by $P_\infty(x, y)$ and a subdominant term associated with the limit of the second derivative of the effective potential at zero field as we have claimed.

In fact, we shall prove the following proposition [see (1.15)]:

$$\lim_{N,n\to\infty} \langle \phi(x_1)\,\phi(x_2)\rangle = P_\infty(x_1, x_2) - \lim_{N,n\to\infty} D^2_{y_1,y_2} V^n(0) \prod_{i=1}^{2} \partial_{\mu_i} G_n(y_i, x_i)$$

where the dominant term $P_\infty(x_1, x_2)$ is given by $b_\infty^{-1} \Delta(x_1, x_2)$ plus a correction with faster falloff than $(1+|x_1-x_2|)^{-(d+2)}$; and the decay of the second term is bounded by $(1+|x_1-x_2|)^{-(d+\varepsilon'')}$, $\varepsilon''>0$ and small.

Differentiating twice $\ln \mathsf{Z}(h)$ of (1.6), we obtain

$$\begin{aligned}\langle \phi(x)\,\phi(y)\rangle = P_n(x, y) + \Bigg\{\int D\phi \exp\left[-\frac{b_n}{2}(\phi, \Delta_n\phi)\right] G_n \partial^\dagger_\mu(x, w) \\ \times \left.\frac{\partial^2 \rho}{\partial\chi_\mu(w)\,\partial\chi_\zeta(z)}\right|_{\chi\equiv\partial M_n\phi} \partial_\zeta G_n(z, y)\Bigg\} \\ \times \left\{\int D\phi \exp\left[-\frac{b_n}{2}(\phi, \Delta_n\phi)\right]\rho\right\}^{-1} \qquad (3.1)\end{aligned}$$

where the sum over repeated indices is always considered, and

$$\rho = \exp[-V^n(\chi)] = \exp[-\tilde{V}^n(\chi)]\exp[-\tfrac{1}{2}(\nabla\chi, K_n\nabla\chi)]$$

[by $\exp(-\tilde{V}^n)$ we understand the expression given in (2.7)]. Note that to obtain (3.1) we have considered $\exp[-V^n(\chi+\partial G_n h)]$ at each step, but since $|\sum_u \partial G_n(u, v)| \leqslant L^n$ (which is proved below), it is always possible to take h sufficiently small so that the size of $\chi+\partial G_n h$ (and the small, large regions) is determined by χ. We emphasize that R_{kNn} and the term associated with $D^2_{y_1 y_2} V^n(0)$ are included in the second term of (3.1), and also that in (3.1) the lattice is unitary, i.e., w, x, y, z above are in $\mathbf{Z}^d$. Therefore, to use the results of Section 2, it is necessary to take χ in $L^{-n}\Lambda_N$ given by $\chi = \nabla\mathscr{A}_n\phi$ (the derivative ∇ also in $L^{-n}\Lambda_N$). Since $\partial M_n\phi(w) = L^{-nd/2}\nabla\mathscr{A}_n\phi(L^{-n}w)$ [which comes from (2.3)], the relation between the derivatives of ρ in different lattices is immediate:

$$\left.\frac{\partial^2\rho}{\partial\chi_\mu(w)\,\partial\chi_\zeta(z)}\right|_{\chi\equiv\partial M_n\phi} = L^{nd}\left.\frac{\partial^2\rho}{\partial\chi_\mu(L^{-n}w)\,\partial\chi_\zeta(L^{-n}z)}\right|_{\chi\equiv\nabla\mathscr{A}_n\phi} \qquad (3.2)$$

At zero step we take $V(\phi)$ as an even analytic functional satisfying the hypotheses $\mathbf{2}_0$–$\mathbf{4}_0$ {taking $g(\phi)\equiv\exp[-V(\phi)]$, g satisfies the hypothesis $\mathbf{1}_0$ for the considered models}. Thus, after $n = N - N_0$ renormalization transformations, we have the formula (3.1) and ρ given by $\mathbf{1}_n$–$\mathbf{4}_n$ [where, using

(3.2), χ is in $L^{-n}\Lambda_N$]. Note that in the final lattice $L^{-(N-N_0)}\Lambda_N$ there is only one block (of side L^{N_0}, which is the scale adopted for the block size). Consequently, $\rho(\chi)$ is given by $g^n(\chi)$ or $\exp[-\frac{1}{2}(\nabla\chi, K_n\nabla\chi) - \tilde{V}^n_\Delta(\chi)]$, never by the product (i.e., one block is only in the large- or in the small-field region, never in both).

Considering the last term in (3.1), the large-field contribution to the correlation function is very small and vanishes as n (and N) goes to infinity. Roughly (by $\mathbf{1}_n$), $|g^n(\chi)| \leqslant \exp[k_n(\phi, \Delta_n\phi)]$ (the same bound following for derivatives of g^n, since it is analytic on a strip). For $k_n < b_n/2$ [which follows for $k < b/2$ at zero step; see ref. 2 and (2.12)], the large contribution is bounded by

$$\frac{\int_D D\phi \exp[-(b_n/2 - k_n)(\phi, \Delta_n\phi)]\, L^{nd} \times L^n \times L^n}{\int D\phi \exp[-\frac{1}{2}b_n(\phi, \Delta_n\phi)]\, \rho}$$

where L^n above is a bound for $|\sum_y \partial G_n(z, y)|$ [using $G_n \simeq c\Delta^{-1}$, which follows from (1.8), (1.13), and (2.15)]. It is easy to show that the denominator is bigger than a positive constant [remember that the measure is concentrated in zero and that $\rho(0) = 1$]. The following inequality is proved in ref. 2:

$$\int_D (\nabla\psi^n)^2 (z')\, dz' = (\phi, \Delta_n\phi)|_{\phi\in D} \geqslant c(n_0 + n)^{2\nu - d^2}, \qquad z' \in L^n\Lambda_N$$

Thus, for $\nu > \frac{1}{2}d^2 + 1$ (as considered here), the large-field region is controlled by $\exp[-c(n_0 + n)^2]$, and so, very small, vanishing rapidly as n goes to infinity.

In the small-field region, $\rho(\chi) \equiv \exp[-\frac{1}{2}(\nabla\chi, K_n\nabla\chi) - \tilde{V}^n_\Delta(\chi)]$, and we have

$$\begin{aligned}\partial^2_{\mu\nu}\rho = &-[\partial^2_{\mu\nu}\tilde{V}^n_\Delta(\chi)]\,\rho + [\partial_\mu \tilde{V}^n_\Delta(\chi)][\partial_\nu \tfrac{1}{2}(\nabla\chi, K_n\nabla\chi)]\,\rho \\ &+ [\partial_\mu \tfrac{1}{2}(\nabla\chi, K_n\nabla\chi)][\partial_\nu \tilde{V}^n_\Delta(\chi)]\,\rho - \partial^2_{\mu\nu}[\tfrac{1}{2}(\nabla\chi, K_n\nabla\chi)]\,\rho \quad (3.3)\end{aligned}$$

where ∂_μ above shall be understood as $\partial/\partial\chi_\mu(L^{-n}\cdot)$.

From $\mathbf{4}_n$, the potential second derivative [first term on the RHS of (3.3)] is bounded by $cL^{-n(d+2)}\delta^{n_0+n}(n_0+n)^{4\nu+4d}$, which makes its contribution to the two-point function smaller than

$$cL^{-n(d+2)}\delta^{n_0+n}(n_0+n)^{4\nu+4d} \times L^{nd}L^nL^n$$

which vanishes as $n \to \infty$ since $\delta < 1$.

The second and third terms on the RHS of (3.3) are limited by

$$cL^{-n(d+1)}\delta^{n_0+n}(n_0+n)^{4\nu+4d} \times cL^{-n(d+1)}(n_0+n)^{2\nu+2d}$$

(due to $\mathbf{4}_n$ and the bounds for the derivatives of K_n proved in the next section) and thus their contribution also vanishes as $n \to \infty$.

A more delicate analysis is necessary for the second derivative of the irrelevant quadratic part [last term in (3.3)]: it corresponds to $D^2_{y_1 y_2} V^n(0)$ in (1.15), and comes into the two-point function (3.1) as

$$\begin{aligned} L^{nd} & \left.\frac{\partial^2[\frac{1}{2}(\nabla\chi, K_n \nabla\chi)]}{\partial\chi_\mu(L^{-n}w)\,\partial\chi_\nu(L^{-n}z)}\right|_{\chi \equiv \nabla\mathscr{A}_n\phi} \partial_\mu G_n(w, x)\,\partial_\nu G_n(z, y) \\ &= L^{nd} \times (L^{-nd})^2 \times (\nabla^\dagger_\alpha K_n^{\alpha\beta\mu\nu} \nabla_\beta)(L^{-n}w, L^{-n}z)\,\partial_\mu G_n(w, x)\,\partial_\nu G_n(z, y) \\ &\equiv \mathscr{I}^n_{22}(x, y) \end{aligned} \tag{3.4}$$

where ∇_β is the derivative in $L^{-n}\Lambda_N$ (and ∂_μ in the unitary lattice). Using the relation $\nabla_\mu(L^{-n}u, L^{-n}v) = L^{n(d+1)}\partial_\mu(u, v)$, we obtain

$$\mathscr{I}^n_{22}(x, y) = L^{-n(d-2)} K_n^{\alpha\beta\mu\nu}(L^{-n}u, L^{-n}v)\,\partial_\alpha\partial_\mu G_n(u, x)\,\partial_\beta\partial_\nu G_n(v, y) \tag{3.5}$$

From Eqs. (3), (13), and (14) of Section 6 of ref. 2,

$$\begin{aligned} K_{n+1}(L^{-(n+1)}u, L^{-(n+1)}v) &= L^{d-2} K_n(L^{-n}u, L^{-n}v) \\ &\quad + \bar{\bar{K}}_{n+1}(L^{-(n+1)}u, L^{-(n+1)}v) \end{aligned} \tag{3.6}$$

where

$$|\bar{\bar{K}}_{n+1}(L^{-(n+1)}u, L^{-(n+1)}v)| \leqslant c\delta^{n_0+n}(n_0+n)^{-2\nu} \exp[-2\alpha L^{-(n+1)}\,|u-v|] \tag{3.7}$$

Starting with local interactions, iterating gives

$$\begin{aligned} & L^{-n(d-2)} K_n(L^{-n}u, L^{-n}v) \\ &\quad = \sum_{j=1}^{n-1} L^{-(n-j)(d-2)} \frac{c\delta^{n_0+(n-j)}}{[n_0+(n-j)]^{2\nu}} \exp(-2\alpha' L^{-n+j}\,|u-v|) \\ &\quad = \sum_{k=1}^{n-1} L^{-k(d-2)} \frac{c\delta^{n_0+k}}{(n_0+k)^{2\nu}} \exp(-2\alpha' L^{-k}\,|u-v|) \end{aligned} \tag{3.8}$$

And with $\delta^k \leqslant cL^{-2k}$ (see next section),

$$\begin{aligned} |L^{-n(d-2)} K_n(L^{-n}u, L^{-n}v)| &\leqslant c \sum_{k=1}^{n-1} L^{-kd} \exp(-2\alpha' L^{-k}\,|u-v|) \\ &\leqslant \frac{c}{(1+|u-v|)^d} \end{aligned} \tag{3.9}$$

Furthermore, from the bounds established in ref. 2 for the kernels of Γ, $\mathscr{A}$, $\nabla\mathscr{A}$, and $\nabla\nabla\mathscr{A}$, we have

$$|\Gamma_k(u, v)| \leqslant c \exp(-\alpha' |u-v|)$$

$$|M_k(u, v)| \leqslant cL^{-k(d-2)/2} \exp(-\alpha' |L^{-k}u - v|)$$

[which leads to $|M_n \Delta_n^{-1} M_n^\dagger(u, v)| \leqslant cL^{-n(d-2)}$, (1.13)], and also

$$|\partial_\mu \partial_\nu M_k(u, v)| < L^{-k(d+2)/2} ck^d \exp(-\alpha' |L^{-k}u - v|)$$

leading to

$$|\partial_\mu \partial_\nu M_k \Gamma_k M_k^\dagger(u, v)| \leqslant ck^d L^{-kd} \exp(-\alpha' L^{-k} |u-v|)$$

And from (1.8), (1.12), (1.13), (2.15),

$$|\partial_\mu \partial_\nu G_n(u, v)| \leqslant \frac{c}{(1+|u-v|)^{d-\varepsilon'}}$$

for ε' small (where $k^d < cL^{\varepsilon' k}$). Thus, we obtain

$$\begin{aligned} |\mathscr{I}_{22}^n(x, y)| &\leqslant c \sum_{u,v} (1+|x-u|)^{-d+\varepsilon'} (1+|u-v|)^{-d} (1+|v-y|)^{-d+\varepsilon'} \\ &\leqslant \frac{c}{(1+|x-y|)^{d-\varepsilon''}} \end{aligned} \tag{3.10}$$

i.e., faster falloff than $\Delta^{-1}(x, y)$.

Finally, we consider the behavior of $P_n(x, y)$, for $n \to \infty$,

$$P_\infty(x, y) = b_\infty^{-1} \Delta(x, y) - b_\infty^{-1} \mathscr{C}_2(x, y) \tag{3.11}$$

where

$$\begin{aligned} |\mathscr{C}_2(x, y)| &\equiv \left| \sum_{k=0}^{\infty} \left(\frac{b_k - b_\infty}{b_k} \right)^2 M_k \Gamma_k M_k^\dagger(x, y) \right| \\ &\leqslant \frac{4}{b_\infty^2} \sum_{k=0}^{\infty} \delta^{2k} |M_k \Gamma_k M_k^\dagger(x, y)| \\ &\leqslant c \frac{4}{b_\infty^2} \sum_{k=0}^{\infty} L^{-4k} L^{-k(d-2)} \exp(-\alpha' L^{-k} |x-y|) \\ &\leqslant c(1+|x-y|)^{-(d+2)} \end{aligned} \tag{3.12}$$

and there is also a faster falloff than $\Delta^{-1}(x, y)$, proving the claim.

We shall emphasize the fine control of subdominant terms as another advantage in using the orthogonal representation: the error term here, $\mathcal{O}(|x|^{-d+\varepsilon})$, is smaller than those obtained in refs. 4 and 5.

4. PROVING HYPOTHESIS 4_n

4.1. Technical Structures

To prove the hypothesis $\mathbf{4}_n$ we adopt the technical procedure presented in ref. 2. The notation (except for the wavefunction constant, which we name b_n) and the results obtained there are used here without repeating the proofs. Essentially, this subsection lists the main technical structures carefully described in ref. 2.

The Gibbs factor representation (2.7) is recovered (i.e., we prove that it is valid for $n+1$) after introducing Mayer and cluster expansions in the renormalization transformation (which controls the nonlocal potentials and interactions between fluctuation fields) and separating the main terms (the terms associated with local interactions and small fluctuation fields). Thus, after several manipulations (all details in ref. 2), writing χ^n as χ, and χ^{n+1} as χ', we have

$$\exp[-W'(\chi')] = \sum_{\{\bar{X}_\zeta\}} \prod_\zeta \rho^{D'}_{\bar{X}_\zeta}(\chi') \times \prod_{\Delta \not\subset D'} \exp[-w'_\Delta(\chi')] \tag{4.1}$$

with w'_Δ associated with the local interactions and small fluctuation fields

$$\exp[-w'_\Delta(\chi')] = \prod_{\Delta' : L^{-1}\Delta' \subset \Delta} \int \exp[-\tilde{V}_{\Delta'}(\chi^0) - \tfrac{1}{2}\delta^2 V_{\Delta'}(\chi^0)] \times 1_0(Z_{\Delta'})\, d\mu_{b^{-1}}(Z_{\Delta'}) \tag{4.2}$$

where Z (fluctuation field) is defined by rewriting the renormalization transformation as

$$\exp[-W'(\chi')] = \int \exp[-V(L^{-d/2}\chi'(L^{-1}\cdot) + \nabla\mathscr{Z}(\cdot))]\, d\mu_{b^{-1}}(Z) \tag{4.3}$$

with

$$W'(\chi') = \tilde{V}'(\chi') + \tfrac{1}{2}\delta^2 W'(\chi') + W'(0) \tag{4.4}$$

(remember that we always separate the local quadratic and constant parts in W'), and

$$\mathscr{Z}^n(z) = (\mathscr{A}_n Q\Gamma_n^{1/2} Z^n)(z) \tag{4.5}$$

where Γ_n is already defined (1.9), and also $\mathscr{A}_n$ in (2.3), (1.11). The operator $Q: \bar{\Lambda}_{N-n} \equiv \Lambda_{N-n} \backslash L\Lambda_{N-n-1} \to \Lambda_{N-n}$ is defined so that $CQ=0$, and does the same as the δ-function in (1.4). In other words,

$$\exp[-W'(\chi')] = \int \exp[-V(L^{-d/2}\chi'(L^{-1}\cdot) + \nabla\mathscr{A}_n\eta)]\,\delta(C\eta)$$
$$\times \exp[-\tfrac{1}{2}b_n(\eta, \Delta_n\eta)]\,D\eta$$

which is the expression used in ref. 1. See Eq. (23) in Section 2 of ref. 2 for details about Q. In (4.2), $Z_{\Delta'}$ means $Z|_{\Delta'}$ and $1_0(Z_{\Delta'})$ is related to the size of Z: to obtain (4.1) a partition of unity specifying the magnitude of Z was introduced,

$$1 = \sum_{\bar{p}} 1_{\bar{p}}(Z) \tag{4.6}$$

where $\bar{p} = (p_x)$, $x \in \bar{\Lambda}_{N-n}$, $p_x = 0, 1,\ldots$, and

$$1_{\bar{p}}(Z) = \prod_{x \in \bar{\Lambda}_{N-n}} 1(B(n_0+n)^\nu\, p_x \leqslant |Z_x| < B(n_0+n)^\nu\,(p_x+1)) \tag{4.7}$$

B a constant. Thus, in (4.2) we are considering only small fluctuation fields, as noted before. A structure to decouple the nonlocal dependence of the fields $\nabla\mathscr{Z}$ and $\nabla\nabla\mathscr{Z}$ on Z has also been introduced by defining the interpolating field $\nabla_\mu\mathscr{Z}^s$:

$$\nabla_\mu\mathscr{Z}^s(z) = \sum_{i<j}\sum_x s_{ij}(1_{U^i}(z)\,1_{U^j}(x) + 1_{U^j}(z)\,1_{U^i}(x))(\nabla_\mu\mathscr{A}Q\Gamma^{1/2})(z,x)\,Z(x)$$
$$+\sum_i\sum_x 1_{U^i}(z)\,1_{U^j}(x)(\nabla_\mu\mathscr{A}Q\Gamma^{1/2})(z,x)\,Z(x) \tag{4.8}$$

where, given $\{X_j\}$, $\{Y_\alpha\}$, and $\{Y_\beta\}$ (sets associated with the large-field region, Mayer expansions of the potential, and its irrelevant quadratic part, respectively), the sets U^i, $i=1,\ldots,n$, are formed by dividing $L^{-n}\Lambda_N$ into components connected with X_j, Y_α, or Y_β [those U^i not intersecting X_j, Y_α, or Y_β are Δ sets—that is the case in (4.2)]. Set

$$\chi^s(\cdot) = L^{-d/2}\chi'(L^{-1}\cdot) + \nabla\mathscr{Z}^s(\cdot) \tag{4.9}$$

χ^0 in (4.2) means that we consider only "local" interactions between fluctuation fields.

The sum in (4.1) runs over the sets of disjoint $\tilde{X}_\zeta$, with $D'\backslash\bigcup_\zeta \tilde{X}_\zeta$, each $D' \cap \tilde{X}_\zeta$ being a union of connected components of D' or empty. To define $\tilde{X}_\zeta$ we introduce $\tilde{X}$, which contains the large-field region, large fluctuations, and nonlocal interactions:

$$\tilde{X} = \tilde{R} \cup (\overline{\textstyle\bigcup_j L^{-1}X_j}) \cup (\overline{\textstyle\bigcup_\alpha L^{-1}Y_\alpha}) \cup (\overline{\textstyle\bigcup_\beta L^{-1}Y_\beta}) \cup (\overline{\textstyle\bigcup_{\gamma:n_\gamma>1} L^{-1}\underline{U}_\gamma}) \tag{4.10}$$

with $|Y_\alpha|, |Y_\beta| > 1$ coming from Mayer expansions of the effective potential and its irrelevant quadratic part; and $\tilde{R}$ defined as $\overline{L^{-1}R}$ together with the next-neighbor blocks Δ in $L^{-(n+1)}\Lambda_N$, where R represents the large-fluctuation fields,

$$R = \bigcup_{x \in \bar{\Lambda}_{N-n}} \{z \in L^{-n}\Lambda_N : d(z, x) < 2\alpha'^{-1} \log(1 + p_x)\} \tag{4.11}$$

p_x is given by (4.7); α' is the same as in (2.4). The sets $\underline{U}_\gamma$ are associated with cluster expansions decoupling the nonlocal dependence of the fluctuation fields $\nabla\mathscr{Z}$ and $\nabla\nabla\mathscr{Z}$ on Z, i.e., related to U^i, U^j given in (4.8). To understand them, we introduce Γ, the subset of pairs $\{(i, j)\}_{i<j}$ (i, j from U^i, U^j) and $\{\Gamma_\gamma\}$, the connected components of Γ. For $i \in \operatorname{supp} \Gamma_\gamma$ (union of vertices i, j in Γ_γ) we denote the corresponding U by U^i_γ, and thus define $\underline{U}_\gamma$ as $\bigcup_i U^i_\gamma$. Finally, $\tilde{X}_\zeta$ is obtained from the partition of $\tilde{X}$ into polymers, i.e., connected components of the graph drawn on the blocks Δ in $\tilde{X}$ formed by lines joining two different blocks if one is in $L^{-1}R$ and the other its next neighbor, or if both are in a single $L^{-1}X_j$, single $L^{-1}Y_\alpha$, single $L^{-1}Y_\beta$, or single $\underline{U}_\gamma$, $n_\gamma > 1$.

The polymer activity in (4.1) is given by

$$\begin{aligned}
\rho^{D'}_{\tilde{X}}(\chi') = &\sum_{\bar{p}, \{X_j\}, \{Y_\alpha\}, \{Y_\beta\}, \{\bar{U}_\gamma\}} \int \prod_\gamma S(\bar{U}_\gamma) \prod_j g^D_{X_j}(\chi^s) \prod_\alpha \{\exp[-\tilde{V}_{Y_\alpha}(\chi^s)] - 1\} \\
&\times \prod_\beta \{\exp[-\tfrac{1}{2}\delta^2 V_{Y_\beta}(\chi^s)] - 1\} \prod_{\Delta \subset L\tilde{X} \setminus \bigcup_j X_j} \exp[-\tilde{V}_\Delta(\chi^s)] \\
&\times \prod_{\Delta \subset L\tilde{X}} \exp[-\tfrac{1}{2}\delta^2 V_\Delta(\chi^s)] \\
&\times 1_{\bar{p}}(Z_{L\tilde{X}})\, d\mu_{b^{-1}}(Z_{L\tilde{X}}) \Big/ \prod_{\Delta \subset \tilde{X} \setminus D'} \exp[-w'_\Delta(\chi')] \\
\equiv &\sum_{\bar{p}, \ldots, \{\bar{U}_\gamma\}} \int \prod S(\bar{U}_\gamma)\, F(X_j, Y_\alpha, Y_\beta; \chi^s) \\
&\times 1_{\bar{p}}(Z_{L\tilde{X}})\, d\mu_{b^{-1}}(Z_{L\tilde{X}}) \Big/ \prod_\Delta \exp[-w'_\Delta(\chi')]
\end{aligned} \tag{4.12}$$

where $\bar{U}_\gamma = \{U^i_\gamma\}_{i=1}^{n_\gamma}$;

$$S(\bar{U}^\gamma) = \sum_{\Gamma_c} \int ds_{\Gamma_c}\, \partial_s^{\Gamma_c} \tag{4.13}$$

with the sum running over all connected graphs Γ on n_γ points; $ds_\Gamma \equiv \prod_{(i,j) \in \Gamma} ds_{ij}$; $\partial^\Gamma_s \equiv \prod_{(i,j)} (\partial/\partial s_{ij})$; and the following restrictions were assumed:

(a) $\bar{p}$ vanishes outside $L\tilde{X}$.

(b) X_j are disjoint, $X_j \cap D$ formed by connected components of $D = L(D' \cup \overline{L^{-1}R})$, $D \cap L\tilde{X} \subset \bigcup_j X_j$.

(c) Y_α, $Y_\beta \subset L\tilde{X}$, $|Y_\alpha|$, $|Y_\beta| > 1$, $Y_\alpha \cap (\bigcup_j X_j) = \varnothing$.

(d) $\{\underline{U}_\gamma\}$ are partitions of $L\tilde{X}$, and U^i_γ is connected with X_j, Y_α, Y_β.

(e) The graph on the blocks $\Delta \subset \tilde{X}$ is connected (constructed following the procedure described above).

In a few words, the nonlocal interactions and the large fields are considered in $\rho^D_{\tilde{X}}$.

The hypothesis $\mathbf{4}_n$ is assumed in the small-field region $D' = \varnothing$, where W' may be taken as $W'(\chi') = \sum_\Delta w'_\Delta(\chi') + \sum_Y W'_Y(\chi')$, with W'_Y given by

$$\exp[-W'_Y(\chi')] = \sum_{\{\tilde{X}_\zeta\}} \prod_\zeta \rho^{\varnothing}_{\tilde{X}_\zeta}(\chi') \tag{4.14}$$

$\tilde{X}_\zeta$ disjoint, $\bigcup_\zeta \tilde{X}_\zeta = Y$. So we write

$$\tilde{V}'_Y = W'_Y - W'_Y(0) - \tfrac{1}{2}\delta^2 W'_Y \qquad (|Y| > 1) \tag{4.15}$$

$$\tilde{V}'_\Delta = W'_\Delta - W'_\Delta(0) - \tfrac{1}{2}\delta^2 W'_\Delta + v'_\Delta \tag{4.16}$$

$$v'_\Delta = w'_\Delta - w'_\Delta(0) - \tfrac{1}{2}\delta^2 w'_\Delta \tag{4.17}$$

The expression (4.14) involving potential exponentiation is suitable for work on derivatives: since it is given by disjoint polymers, the derivative will not cause a proliferation of terms (which would happen due to the product). Thus, in the expression

$$\frac{\partial}{\partial\chi'(z)} \exp[-W'_Y(\chi')] = \frac{\partial}{\partial\chi'(z)} \left\{ \sum_{\{\tilde{X}_\zeta\}} \prod_\zeta \rho^{\varnothing}_{\tilde{X}_\zeta} \right\} \tag{4.18}$$

only the term $\rho^{\varnothing}_{\tilde{X}_\zeta}$ such that $z \in \tilde{X}_\zeta$ is changed by the derivative operator. Concerning the relation between the derivatives of W'_Y and of $\exp(-W'_Y)$, it is obvious that

$$-D_z W'_Y = D_z \ln \exp(-W'_Y) = \frac{D_z \exp(-W'_Y)}{\exp(-W'_Y)} \tag{4.19}$$

$$D^2_{zz'} W'_Y = \frac{D^2_{zz'} \exp(-W'_Y)}{\exp(-W'_Y)} - \frac{D_z \exp(-W'_Y) \times D_{z'} \exp(-W'_Y)}{\exp(-2W'_Y)} \tag{4.20}$$

where we wrote $\partial/\partial\chi'(z)$ as D_z and $\partial^2/\partial\chi'(z)\,\partial\chi'(z')$ as $D^2_{zz'}$.

4.2. The Local Potential Estimates

We first study the derivatives of v'_Δ, the potential part which includes local interactions and small fluctuation fields. Here, the simplicity of the renormalization transformation (4.2) makes it easy to see the extra factors L^{-1} obtained with the derivatives (hypothesis $\mathbf{4}_n$). To make this transparent, we present the analysis in detail (and repeat some arguments described in ref. 2).

Equations (4.2) and (4.17) give us v'_Δ in terms of the previous scale. In Appendix 1 of ref. 2 it is proved that

$$\begin{aligned} |\nabla_\mu \mathscr{Z}^s(z)| &< \tfrac{1}{2}(n_0+n)^\nu \\ |\nabla_\mu \nabla_\zeta \mathscr{Z}^s(z)| &< cn^d(n_0+n)^\nu \end{aligned} \tag{4.21}$$

for $z \notin R$; z and $z+L^{-n}e_\mu \in U^i$; $0 \leqslant s \leqslant 1$. Due to (4.21) and hypothesis $\mathbf{2}_n$, the integrand $\exp[-\tilde{V}_{\Delta'}(\chi^0) - \frac{1}{2}\delta^2 V_{\Delta'}(\chi^0)]$ is analytic in χ' [where $\chi^0(\cdot) = L^{-d/2}\chi'(L^{-1}\cdot) + \nabla\mathscr{Z}^0(\cdot)$] on $L^{d/2}\mathscr{K}_{n+1}(\Delta)$ [considering Z on the support of $1_0(Z_{\Delta'})$]. In addition,

$$|\tilde{V}_{\Delta'}(\chi^0)| \leqslant \delta^{n_0+n}, \qquad \tfrac{1}{2}|\delta^2 V_{\Delta'}(\chi^0)| \leqslant c\delta^{n_0+n}(n_0+n)^{2\nu+2d} \tag{4.22}$$

where the last inequality is due to (4.21) and $\mathbf{3}_n$ (later in this section, a bound associated with the irrelevant quadratic term is shown in detail). Thus, on $L^{d/2}\mathscr{K}_{n+1}(\Delta)$,

$$|\exp[-\tilde{V}_{\Delta'}(\chi^0) - \tfrac{1}{2}\delta^2 V_{\Delta'}(\chi^0)] - 1)| \leqslant c\delta^{n_0+n}(n_0+n)^{2\nu+2d} \tag{4.23}$$

leading to

$$|\exp[-w'_\Delta(\chi')] - 1| \leqslant c\delta^{n_0+n}(n_0+n)^{2\nu+2d} \tag{4.24}$$

since

$$\left|\int (1_0(Z_{\Delta'}) - 1)\, d\mu_{b^{-1}}(Z_{\Delta'})\right| \leqslant \exp[-cB^2(n_0+n)^{2\nu}]$$

Hence, w'_Δ is analytic on the region $L^{d/2}\mathscr{K}_{n+1}(\Delta)$ and there

$$|w'_\Delta| \leqslant c\delta^{n_0+n}(n_0+n)^{2\nu+2d} \tag{4.25}$$

To analyze v'_Δ (quartic and higher parts of w'_Δ) we introduce a truncated expansion

$$\begin{aligned} w'_\Delta(\chi') &= \sum_{\Delta' : L^{-1}\Delta' \subset \Delta} -\log \int \exp[-\tilde{V}_{\Delta'}(\chi^0) - \tfrac{1}{2}\delta^2 V_{\Delta'}(\chi^0)]\, 1_0(Z_{\Delta'})\, d\mu_{b^{-1}}(Z_{\Delta'}) \\ &= \sum_{\Delta' : L^{-1}\Delta' \subset \Delta} \langle \mathscr{V}_{\Delta'} \rangle_0^T - \tfrac{1}{2}\langle \mathscr{V}_{\Delta'}; \mathscr{V}_{\Delta'} \rangle_0^T + \tfrac{1}{2}\int_0^1 dt (1-t)^2 \langle \mathscr{V}_{\Delta'}; \mathscr{V}_{\Delta'}; \mathscr{V}_{\Delta'} \rangle_t^T \end{aligned} \tag{4.26}$$

where

$$\mathscr{V}_{\Delta'} \equiv \tilde{V}_{\Delta'}(\chi^0) + \frac{1}{2}\delta^2 V_{\Delta'}(\chi^0)$$

$$\langle * \rangle_t \equiv \frac{\int * \exp[-t\mathscr{V}_{\Delta'}(\chi^0)]\, 1_0(Z_{\Delta'})\, d\mu_{b^{-1}}(Z_{\Delta'})}{\int \exp[-t\mathscr{V}_{\Delta'}(\chi^0)]\, 1_0(Z_{\Delta'})\, d\mu_{b^{-1}}(Z_{\Delta'})} \tag{4.27}$$

and $\langle *;...;*\rangle_t^T$ means the truncated average.

Introducing the complex variable $\bar{z}$ in $w'_\Delta(\bar{z}\chi')$, using the Cauchy integral formula and (4.22), we obtain for $\chi' \in 2\mathscr{K}_{n+1}(\Delta)$ and $m > 0$,

$$\left| \frac{d^m}{d\bar{z}^m}\right|_{\bar{z}=0} \tilde{V}_{\Delta'}(\chi^0(Z)) \Bigg| \leqslant m! \left(\frac{1}{2}L^{d/2}\right)^{-m} \delta^{n_0+n} \tag{4.28}$$

and for $m = 1, 2,$

$$\left| \frac{d^m}{d\bar{z}^m}\right|_{\bar{z}=0} \frac{1}{2}\delta^2 V_{\Delta'}(\chi^0(Z)) \Bigg| \leqslant c\delta^{n_0+n}(n_0+n)^{2\nu+2d} \tag{4.29}$$

(vanishing for $m > 2$).

Thus, using the bounds (4.28), (4.29) above and the truncated expression (4.26), for $\chi' \in 2\mathscr{K}_{n+1}(\Delta)$, $m > 2$,

$$\left| \frac{d^m}{d\bar{z}^m}\right|_{\bar{z}=0} w'_\Delta(\bar{z}\chi') \Bigg| \leqslant L^d m! \left(\frac{1}{4}L^{d/2}\right)^{-m} \delta^{n_0+n} \tag{4.30}$$

and so, for $v'_\Delta(\chi')$ on $2\mathscr{K}_{n+1}(\Delta)$,

$$\begin{aligned} |v'_\Delta(\chi')| &= \left| \sum_{m=4}^{\infty} \frac{1}{m!} \frac{d^m}{d\bar{z}^m}\right|_{\bar{z}=0} w'_\Delta(\bar{z}\chi') \Bigg| \\ &\leqslant 4^4 L^{-d}(1-4L^{-d/2})^{-1}\,\delta^{n_0+n} \leqslant \frac{1}{2}\delta^{n_0+n+1} \end{aligned} \tag{4.31}$$

(large L and $L^{-d} < \delta$).

To study the derivatives of v'_Δ, we take the truncated expression (4.26) and, in w'_Δ, consider the terms of quartic and superior order. For $\partial v'_\Delta(\chi')/\partial\chi'(u)$ we analyze

$$\begin{aligned} \frac{\partial w'_\Delta(\chi')}{\partial \chi'(u)} &= \left\langle \frac{\partial \mathscr{V}_{\Delta'}(\chi)}{\partial \chi'(u)} \right\rangle_0 \\ &\quad - \frac{1}{2}\frac{\partial}{\partial\chi'(u)}\left\{ \langle \mathscr{V}_{\Delta'};\mathscr{V}_{\Delta'}\rangle_0^T - \frac{1}{2}\int_0^1 dt(1-t)^2\, \langle \mathscr{V}_{\Delta'};\mathscr{V}_{\Delta'};\mathscr{V}_{\Delta'}\rangle_t^T \right\} \end{aligned} \tag{4.32}$$

The main part is given by $\partial \tilde{V}_{\Delta'}(\chi)/\partial\chi'(u)$ in the first term on the RHS of (4.32), which is bounded by [for χ' on $L^{d/2}\mathscr{K}_{n+1}(\Delta)$]

$$L^{-d/2} \times cL^{-n(d+1)}\delta^{n_0+n}(n_0+n)^{4\nu+4d}$$

where the factor $L^{-d/2}$ comes from the relation between χ' and χ, and the other from hypothesis $\mathbf{4}_n$. Note that in $\partial \tilde{V}_{\Delta'}/\partial\chi'(u)$ just one set Δ' is considered: the one which contains u (although, given Δ, there exist L^d sets Δ' such that $L^{-1}\Delta' \subset \Delta$). Restricting χ' to $2\mathscr{K}_{n+1}(\Delta)$ we get the factor $(\frac{1}{2}L^{-d/2})^4$ due to Cauchy estimates, and the bound becomes (for $L^{-d} \leqslant \delta$), say,

$$\tfrac{1}{6}c(n_0+n+1)^{4\nu+4d}L^{-(n+1)(d+1)}\delta^{n_0+n+1} \tag{4.33}$$

The second part of the RHS of (4.32) includes terms such as $\partial\delta^2 V_{\Delta'}/\partial\chi'(u)$, which is bounded by

$$L^{-nd} \times L^{-d/2} \times \int_{\Delta'} \nabla K_n(u, z)\, \nabla\nabla \mathscr{X}^s(z)\, dz$$

(the term $\nabla K_n \nabla\nabla\chi'$ is not considered because it disappears within the truncated expansion). To bound K_n we introduce Eq. (3.8) in the integral above, obtaining

$$\left| \int_{\Delta'} \nabla K_n(u, z)\, dz \right| \leqslant L^n \times \frac{c\delta^{n_0+n}}{(n_0+n)^{2\nu}}$$

Now we remark that the size of δ is limited by relations with L, and the main restriction is given by hypothesis 3_n: to obtain (2.14) at step $n+1$ from (3.6) and (3.7) we need $L^{-2}/(n+1) \leqslant \delta/[(n+1)+1]$. Consequently, it is possible to get one constant c which does not depend on n, such that $\delta^n \leqslant cL^{-2n}$ (note that $\lim_{n\to\infty}\{[(n+1)+1]/(n+1)\}^n = e$). Hence,

$$\left| \int_{\Delta'} \nabla K_n(u, z)\, dz \right| \leqslant L^n \times L^{-2n}c \tag{4.34}$$

and the bound for $\partial\delta^2 V_{\Delta'}/\partial\chi'(u)$ becomes

$$\begin{aligned} &L^{-nd} \times L^{-d/2} \times L^n \times L^{-2n} \times c \times \sup(\nabla\nabla\mathscr{X}^s) \\ &\quad \leqslant cL^{-n(d+1)}L^{-d/2} \times (n_0+n)^{2\nu+2d} \end{aligned}$$

[from (4.21)]. But obviously, for v'_Δ (i.e., quartic and superior terms of w'_Δ) we must have another $\delta^2 V_{\Delta'}$ or $\tilde{V}_{\Delta'}$ together with the term K_n in the considered part of the expansion (4.32), which is then limited by

$$\delta^{n_0+n}(n_0+n)^{2\nu+2d} \times L^{-n(d+1)} L^{-d/2}(n_0+n)^{2\nu+2d}\,\text{const}$$
$$\leqslant \frac{c}{6}\,\delta^{n_0+n+1} L^{-(n+1)(d+1)}(n_0+n)^{4\nu+4d}$$

(note that c, although depending on L, does not depend on n).

The same bound follows for the last term of (4.32), leading to

$$\left|\frac{\partial v'_\Delta(\chi')}{\partial\chi'(u)}\right| \leqslant \frac{1}{2}\,cL^{-(n+1)(d+1)}(n_0+n+1)^{4\nu+4d}\,\delta^{n_0+n+1} \tag{4.35}$$

Concerning $\partial^2 v'/\partial\chi'(u)\,\partial\chi'(r)$, we shall analyze an expression similar to (4.32). For the first part $\langle \partial^2\mathscr{V}_{\Delta'}(\chi')/\partial\chi'(u)\,\partial\chi'(r)\rangle_0^T$ we obtain the bound

$$L^{-d/2}L^{-d/2} \times L^{-n(d+2)}\delta^{n_0+n}c(n_0+n)^{4\nu+4d}$$

for $\chi' \in L^{d/2}\mathscr{K}_{n+1}(\Delta)$, and restricting to $\chi' \in 2\mathscr{K}_{n+1}(\Delta)$,

$$\frac{c}{6}\,L^{-(n+1)(d+2)}(n_0+n+1)^{4\nu+4d}\,\delta^{n_0+n+1}$$

Considering the rest of the expression, we note that $\partial^2\delta^2 V_{\Delta'}/\partial\chi'(u)\,\partial\chi'(r)$ disappears in $\langle\cdot;\cdot\rangle_0^T$ and also in $\int dt(1-t)^2\,\langle\cdot;\cdot;\cdot\rangle_t^T$, and that the greatest term is associated with

$$\left\langle \frac{\partial\delta^2 V_{\Delta'}(\chi)}{\partial\chi'(u)};\frac{\partial\delta^2 V_{\Delta'}(\chi)}{\partial\chi'(r)}\right\rangle_0^T$$

which is bounded by

$$(L^{-nd}L^{-d/2}L^{n}L^{-2n}c \times \sup(\nabla\nabla\mathscr{L}^s))^2$$
$$\leqslant \frac{c}{6}\,L^{-(n+1)(d+2)}(n_0+n+1)^{4\nu+4d}\,\delta^{n_0+n+1}$$

(for $L^{-d} \leqslant \delta$). Altogether, we get

$$\left|\frac{\partial^2 v'_\Delta(\chi')}{\partial\chi'(u)\,\partial\chi'(r)}\right| \leqslant \frac{1}{2}\,cL^{-(n+1)(d+2)}(n_0+n+1)^{4\nu+4d}\,\delta^{n_0+n+1} \tag{4.36}$$

4.3. The Polymer Estimates

Now, to estimate the derivatives of W'_Y (the terms of order four and higher, since we are interested in $\tilde{V}'_Y$) we shall evaluate the polymer activities (4.12).

In Section 5 of ref. 2 a general polymer is carefully analyzed and bounds are established for the activities. In the small-field region $D' = \varnothing$, several considerations are made to improve the bounds and to limit $|W'_Y|$ properly, thus proving hypothesis $\mathbf{2}_n$, i.e., $|\tilde{V}^n_Y| \leqslant \delta^{n_0+n} \exp[-2\alpha' \mathscr{L}(Y)]$.

Here we use all these results without repeating the proofs. We show how to extract the desired additional factors L^{-n} in the polymer expansion for the derivatives of $\tilde{V}'_Y$ as compared to the expansion for $\tilde{V}'_Y$ (already studied and controlled in ref. 2). Briefly, in the expression for DW'_Y, (4.19) (discarding in W'_Y the terms up to second order in χ'), we show that it is always possible to make explicit one part bounded by

$$(n_0 + n + 1)^{4\nu + 4d} L^{-(n+1)(d+1)} \delta^{n_0+n+1} \exp[-2\alpha' \mathscr{L}(Y)]$$

and write the total expression as this part times the rest, which is limited by, say,

$$\{\exp[|W'_Y - W'_Y(0)|]\}^2 \leqslant \exp(\delta^{n_0+n})$$

Similar considerations follow for $D^2 W'_Y$, (4.20).

Let us start with one derivative in W'_Y, analyzing (4.18) and (4.19), i.e., the factors $\partial \rho^{\varnothing}_{\tilde{X}} / \partial \chi'(z)$ therein.

Polymers with $R \neq \varnothing$ (containing large fluctuation fields) involve factors such as $\exp[-cB^2(n_0+n)^{2\nu} |\bar{R}|]$ [see, e.g., Eq. (43) in Section 5 of ref. 2], much smaller than the factors we need. Let us then consider those with $R = \varnothing$.

Terms with the derivative $\partial/\partial\chi'(z)$ acting in one Y_β [see (4.12)] have $\partial \delta^2 V_{Y_\beta} / \partial \chi'(z)$ bounded by

$$L^{-nd} \times L^{-d/2} \times \int_{Y_\beta} \nabla K(z, z') \nabla\nabla \mathscr{X}^s(z') \, dz' \tag{4.37}$$

[remembering that L^{-nd} is due to the lattice $L^{-n}\Lambda_N$, and $L^{-d/2}$ due to $\delta^2 V_{Y_\beta} \equiv \delta^2 V_{Y_\beta}(\chi^s)$, with $\chi^s = L^{-d/2}\chi' + \nabla \mathscr{X}^s$]. Note that the derivatives of terms $\delta^2 V_{Y_\beta}$, such as $(\nabla\chi', K\nabla\chi')$, contribute just to the quadratic part of W' since

$$\begin{aligned}
\exp[-W'(\chi')] &= \int \exp\left[-\sum_Y \tilde{V}_Y(\chi) - \tfrac{1}{2}\delta^2 V_Y(\chi)\right] 1|_{R=\varnothing}(Z) \, d\mu_{b^{-1}}(Z) \\
\Rightarrow W'(\chi') &= \left\langle \sum_Y \tfrac{1}{2}\delta^2 V_Y + \tilde{V}_Y \right\rangle \\
&\quad - \tfrac{1}{2} \left\langle \sum_Y \tfrac{1}{2}\delta^2 V_Y + \tilde{V}_Y ; \sum_Y \tfrac{1}{2}\delta^2 V_Y + \tilde{V}_Y \right\rangle_0^T + \text{rest}
\end{aligned} \tag{4.38}$$

[see (4.3) and (4.26)], showing that the constant terms on Z disappear in the truncated expansion (except for the first term where $\delta^2 V_Y$ is related to the quadratic part of W', which we do not consider here). From (4.21) and (4.34), (4.37) becomes

$$\leqslant cL^{-n(d+1)}L^{-d/2}(n_0+n)^{2\nu+2d} \tag{4.39}$$

But, obviously, the expansion for the derivative of quartic and higher parts of W'_Y must contain other terms $\delta^2 V_{Y_\beta}$, $\tilde{V}_{Y_\alpha}$,... (never one unique $\delta^2 V_{Y_\beta}$). This other term is bounded by, say,

$$c\delta^{n_0+n}(n_0+n)^{2\nu+2d}\exp[-2\alpha'\mathscr{L}(Y_\alpha)]\,G^{-|Y_\alpha|}, \qquad G \text{ large}$$

or

$$c\delta^{n_0+n}(n_0+n)^{2\nu+2d}\exp[-7\alpha'\mathscr{L}(Y_\beta)]$$

[see Eqs. (55) and (57) of Section 5 of ref. 2]. Thus, considering the product, we obtain the factor

$$cL^{-(n+1)(d+1)}\delta^{n_0+n+1}\exp[-2\alpha'\mathscr{L}(\tilde{X})](n_0+n+1)^{4\nu+4d}$$

(for the polymer in $\tilde{X}$).

The term with just one Y_α and $\mathscr{L}(Y_\alpha) > \mathscr{L}(\tilde{X})$ yields

$$\begin{aligned} &L^{-d/2}\times(L^{-(d/2+1)}\delta)\times L^{-n(d+1)}\delta^{n_0+n} \\ &\quad\times\exp[-2\alpha'\mathscr{L}(\tilde{X})](n_0+n)^{4\nu+4d}\,G^{-|\tilde{X}|} \\ &\quad= L^{-(n+1)(d+1)}\delta^{n_0+n+1}\exp[-2\alpha'\mathscr{L}(\tilde{X})](n_0+n)^{4\nu+4d}\,G^{-|\tilde{X}|} \end{aligned}$$

where $(L^{-(d/2+1)}\delta)$ is extracted from the difference between $\mathscr{L}(Y_\alpha)$ and $\mathscr{L}(\tilde{X})$ (L^{N_0} large), and the other factors come from the relation between χ^s and χ', and hypothesis $\mathbf{4}_n$ (assuming step n to prove $n+1$). With one Y_α and $\mathscr{L}(Y_\alpha) = \mathscr{L}(\tilde{X})$ we have

$$2\delta^{n_0+n}\exp[-2\alpha'\mathscr{L}(\tilde{X})]\times L^{-n(d+1)}\times L^{-d/2} \tag{4.40}$$

where the first factor above is a bound for the polymer activity with the field χ' in the region $\frac{1}{4}L^{d/2}\mathscr{K}_{n+1}(\tilde{X})$; the second is due to the hypothesis $\mathbf{4}_n$; and the last is due to the relation between χ^s and χ'.

We note that the estimate of similar polymers in the proof of hypothesis $\mathbf{2}_n$ in Section 5 of ref. 2 contains an extra factor L^{d-1} as compared to the first term in (4.40) above. The fact is that in Section 5 of ref. 2 it is necessary to sum over all Y_α such that $\overline{L^{-1}Y_\alpha} = \tilde{X}$, and there may exist up to L^{d-1} sets Y_α for each $\tilde{X}$. But in our case (for the derivative) we consider just one set since only one of these L^{d-1} different sets Y_α may contain the site z associated with the derivative $\partial/\partial\chi'(z)$.

The restriction to the region $\chi' \in 2\mathscr{K}_{n+1}(\tilde{X})$ introduces a factor $(8L^{-d/2})^4$ leading in (4.40) to $3\delta^{n_0+n+1}L^{-(n+1)(d+1)}\exp[-2\alpha'\mathscr{L}(\tilde{X})]$ (for $L^{-d} < \delta$).

Due to $S(U_\gamma)$ [see (4.12) and (4.13)], the terms with no Y_α or Y_β must contain derivatives of $\exp[-\tilde{V}_\Delta(\chi^s)]$ or $\exp[-\frac{1}{2}\delta^2 V_\Delta(\chi^s)]$. Noting that $\partial_s \exp[-\tilde{V}_\Delta(\chi^s)] = \partial_s(\exp[-\tilde{V}_\Delta(\chi^s)] - 1)$ [∂_s as defined in (4.13)], and the same for $\exp[-\frac{1}{2}\delta^2 V_\Delta(\chi^s)]$, we follow an analysis similar to those described for $\tilde{V}_{Y_\alpha}$ and $\delta^2 V_{Y_\beta}$.

Thus, from (4.18), (4.19), the estimates, and the comments above, we get

$$\left|\frac{\partial \tilde{V}'_Y(\chi')}{\partial\chi'(z)}\right| \leqslant c(n_0+n+1)^{4\nu+4d} L^{-(n+1)(d+1)}\delta^{n_0+n+1}\exp[-2\alpha'\mathscr{L}(Y)]$$

Now we turn to the derivation of the second derivative assumption. From (4.14)

$$\frac{\partial^2 \exp[-W'_Y(\chi')]}{\partial\chi'(z)\,\partial\chi'(z')} = \frac{\partial^2}{\partial\chi'(z)\,\partial\chi'(z')}\left\{\sum_{\{\tilde{X}_\zeta\}}\prod_\zeta \rho^{\varnothing}_{\tilde{X}_\zeta}(\chi')\right\} \tag{4.41}$$

Let us first remark that, from the truncated expansion for W' [see (4.38) and remarks], the terms including $\partial^2\delta^2 V_Y$ (i.e., the second derivative of the irrelevant quadratic part) contribute only to the derivative of the quadratic part of W' (which does not interest us).

We consider the terms with $R = \varnothing$. Those with just one Y_β, i.e., $\partial^2\delta^2 V_Y$, are discarded in our analysis as described above. For those with two Y_β, one derivative in Y_{β_1} and another in Y_{β_2}, using (4.39), we get

$$\left|\frac{\partial}{\partial\chi'(z)}\delta^2 V_{Y_{\beta_1}}(\chi')\right| \leqslant cL^{-d/2}L^{-n(d+1)}(n_0+n)^{2\nu+2d}$$

and another similar factor due to Y_{β_2}, leading to

$$L^{-(n+1)(d+2)}\delta^{n_0+n+1}c(n_0+n+1)^{4\nu+4d}\exp[-2\alpha'\mathscr{L}(\tilde{X})]$$

[for $L^{-nd}L^d \leqslant c\ \exp[-2\alpha'\mathscr{L}(\tilde{X})]\,\delta^n$, which essentially means $L^{-d} < \delta$; note that for $\tilde{X}$ formed by Y_{β_1} and Y_{β_2}, $\mathscr{L}(\tilde{X})$ is not large, since $|Y_\beta| \leqslant 4$; see the comments below Eq. (14) in Section 3 of ref. 2]. An analysis similar to that already described holds for the terms with one Y_α and one Y_β; with two Y_α; with just one Y_α and $\mathscr{L}(Y_\alpha) > \mathscr{L}(\tilde{X})$; with one Y_α and $\mathscr{L}(Y_\alpha) = \mathscr{L}(\tilde{X})$; and also for those with no Y_α or Y_β. Thus, we obtain

$$\left|\frac{\partial^2 \tilde{V}'_Y}{\partial \chi'(z)\, \partial \chi'(z')}\right| \leqslant cL^{-(n+1)(d+2)}(n_0+n+1)^{4\nu+4d}\,\delta^{n_0+n+1} \exp[-2\alpha' \mathscr{L}(Y)]$$

for $\chi' \in 2\mathscr{K}_{n+1}(Y)$.

5. GENERAL CORRELATIONS AND FINAL COMMENTS

The same procedure adopted in the previous sections to control the two-point function applies for a general correlation. Here, we describe this analysis for the k-point function without the technical details, arguing to show the main points.

Differentiating k times $\ln \mathrm{Z}(h)$ of (1.6), we obtain

$$\begin{aligned}
&\langle \phi(x_1) \cdots \phi(x_k)\rangle^T \\
&\quad = \left\{ \int D\phi \exp\left[-\frac{b_n}{2}(\phi, \Delta_n \phi)\right] \frac{\partial^k \rho}{\partial \chi_{\mu_1}(u_1) \cdots \partial \chi_{\mu_k}(u_k)}\bigg|_{\chi \equiv \partial M_n \phi} \right. \\
&\qquad \left. \times\, \partial_{\mu_1} G_n(u_1, x_1) \cdots \partial_{\mu_k} G_n(u_k, x_k) \right\} \\
&\qquad \times \left\{ \int D\phi \exp\left[-\frac{b_n}{2}(\phi, \Delta_n \phi)\right] \rho \right\}^{-1}
\end{aligned} \tag{5.1}$$

[see (3.1) for details], u_i, x_i now on the unity lattice.

Let us consider only the delicate part: the small-field region with all the derivatives in the same term. Taking the kth derivative of the potential in (5.1), we need to control

$$L^{(nd/2)k} \frac{\partial^k \tilde{V}}{\partial \chi_{\mu_1}(L^{-n}u_1) \cdots \partial \chi_{\mu_k}(L^{-n}u_k)}\bigg|_{\chi \equiv \nabla \mathscr{A}_n \phi} \partial_{\mu_1} G_n(u_1, x_1) \cdots \partial_{\mu_k} G_n(u_k, x_k) \tag{5.2}$$

For the $\tilde{V}$ parts of order $k+2$ and up (in χ) we shall obtain a bound by generalizing hypothesis $\mathbf{4}_n$,

$$|D^k \tilde{V}_Y(\chi)| \leqslant cL^{-n(d/2)k} L^{-nk} \delta^{n_0+n} \exp[-2\alpha' \mathscr{L}(Y)] \tag{5.3}$$

where $\chi \in \mathscr{K}_n(Y)$, $D^k \equiv \partial^k / \partial\chi(z_1) \cdots \partial\chi(z_k)$ [for $k \geqslant 4$, extra factors L^{-n} control the factor $(n_0+n)^{\nu+d}$ which appears in (2.16) and (2.17), $k=2$].

As a rapid argument we examine the renormalization transformation for the local potential. From (4.2), approximately,

$$\exp[-v'_\Delta(\chi')] \simeq c \prod_{\Delta': L^{-1}\Delta' \subset \Delta} \int \exp[-v_{\Delta'}(\chi^0)]\, 1_0(Z_{\Delta'})\, d\mu_{b^{-1}}(Z_{\Delta'}) \qquad (5.4)$$

or, considering only the first order in v,

$$v'_\Delta(\chi') \simeq \sum_{\Delta': L^{-1}\Delta' \subset \Delta} \langle v_{\Delta'}(\chi^0) \rangle \qquad (5.5)$$

where $\langle * \rangle \equiv \int * \, 1_0(Z_{\Delta'})\, d\mu_{b^{-1}}(Z_{\Delta'})$. Writing $D'^k \equiv \partial^k / \partial\chi'(z_1) \cdots \partial\chi'(z_k)$, and for $\chi' \in L^{d/2}\mathscr{K}_{n+1}(\Delta)$,

$$D'^k v'_\Delta(\chi') \simeq (L^{-d/2})^k \langle D^k v_{\Delta'}(\chi^0) \rangle$$
$$|D'^k v'_\Delta(\chi')| \leqslant (L^{-d/2})^k (L^{-n(d/2)k} L^{-nk} \delta^{n_0+n})$$

where $(L^{-d/2})^k$ comes from the relation $\chi' = L^{-d/2}\chi + \nabla\mathscr{Z}$, and the other factor from the assumption (5.3) assumed in the nth step [remarking that the L^d terms from the sum over Δ' in (5.5) disappear with the derivative]. Introducing the complex variable $\bar{z}$ as in Eq. (4.28), using Cauchy estimates, restricting χ' to $2\mathscr{K}_{n+1}(\Delta)$, and considering the potential parts of order $k+2$ and higher (in χ'), we get

$$|D'^k v'_\Delta(\chi')| \leqslant (L^{-d/2})^k L^{-n(d/2)k} L^{-nk} \delta^{n_0+n} \sum_{j=k+2}^{\infty} (\tfrac{1}{4} L^{d/2})^{-j}$$
$$\leqslant L^{-(n+1)(d/2)k} L^{-(n+1)k} \delta^{n_0+n+1}$$

(for $L^{-d} < \delta$), which approximately justifies the generalization proposed in (5.3).

Thus, with (5.3) and $|\sum_{x \in \Lambda_N} \partial_\mu G_n(u, x)| \leqslant L^n$, it follows immediately that the contribution to the k-point truncated function of the potential parts with order higher than k vanishes at the thermodynamic limit [in (1.15), this means $\lim_{N,n \to \infty} R_{kNn} = 0$].

Now we turn to the analysis of the $\tilde{V}$ part with k fields, which remains in the thermodynamic limit. For the kernel S_n of this term, i.e.,

$$S_n(L^{-n}x_1, ..., L^{-n}x_k) \equiv \left. \frac{\partial^k \tilde{V}}{\partial\chi_{\mu_1}(L^{-n}u_1) \cdots \partial\chi_{\mu_k}(L^{-n}u_k)} \right|_{\chi \equiv \nabla\mathscr{A}_n\phi} (0)$$

following the basic property of the renormalization group, which says that the transformation maintains local potential modulo exponentially decaying tails, we expect

$$S_{n+1}(L^{-(n+1)}x_1, ..., L^{-(n+1)}x_k) = (L^{d/2})^k S_n(L^{-n}x_1, ..., L^{-n}x_k) + \mathscr{C}_n(L^{-n}x_1, ..., L^{-n}x_k) \qquad (5.6)$$

where $x_1,\ldots, x_k$ are in the unitary lattice, $L^{d/2}$ is the scaling factor ($\chi' = L^{-d/2}\chi + \cdots$), and, as claimed,

$$|\mathscr{C}_n(L^{-n}x_1,\ldots, L^{-n}x_k)| \leqslant c\delta^{n_0+n} \exp[-cd(L^{-n}x_1,\ldots, L^{-n}x_k)] \quad (5.7)$$

where $d(\cdot,\ldots,\cdot)$ gives, in some sense, the distance between the points (e.g., the length of the shortest connected graph), and δ^{n_0+n} is directly related to the bound for the perturbative potential (hypothesis $\mathbf{2}_n$).

Iterating (5.6) and using (5.7), we obtain

$$\begin{aligned} |L^{-n(d/2)k}S_n(L^{-n}x_1,\ldots, L^{-n}x_k)| &\leqslant \sum_{j=1}^{n} L^{-jdk/2}c\delta^{n_0+j} \exp[-cL^{-j}d(x_1,\ldots, x_k)] \\ &\leqslant c(1+d(x_1,\ldots, x_k))^{-(dk/2+2)} \end{aligned} \quad (5.8)$$

See (3.8) and (3.9). And with $|\partial_\mu G_n(u, x)| \leqslant c(1+|u-x|)^{-d+1}$ [see comments below Eq. (3.9)], we get for the term of the k-point truncated function which survives in the thermodynamic limit (and $n \to \infty$)

$$\begin{aligned} &\left| D^k_{y_1\cdots y_k} V^n(0) \prod_{i=1}^{k} \partial_{\mu_i} G_n(y_i, x_i) \right| \\ &\quad \leqslant \sum_{y_1,\ldots, y_k \in \Lambda_N} c \left\{ \prod_{i=1}^{k} \frac{1}{(1+|x_i-y_i|)^{d-1}} \right\} \frac{1}{(1+d(y_1,\ldots, y_k))^{dk/2+2}} \end{aligned} \quad (5.9)$$

Note that this expression leads to a tree graph decay. It is easy to see that it vanishes in the scaling limit: taking the four-point function as an example, and roughly using

$$\frac{1}{(1+d(y_1,\ldots, y_4))^{dk/2+2}} \leqslant \sum_{\text{pairings}} \frac{c}{(1+|y_1-y_2|)^{d+\varepsilon}(1+|y_3-y_4|)^{d+\varepsilon}}$$

it follows that (5.9) with $k=4$ is bounded by

$$\sum_{\text{pairings}} \frac{c}{(1+|x_1-x_2|)^{d-2+\varepsilon}(1+|x_3-x_4|)^{d-2+\varepsilon}}$$

a slight improvement for the estimatives presented in ref. 4 [although a much better bound may be obtained from (5.9)].

We remark that it is possible to improve the exponent $(dk/2+2)$ in (5.9). Separating the potential $\tilde{V}$ at the zero renormalization step into several parts, $\tilde{V}_{(4)},\ldots, \tilde{V}_{(k)}$ and $\tilde{V}_{(>k)}$ (i.e., the parts with four fields,..., k fields, and more than k fields, respectively), and following each one by the renormalization flow, hypothesis $\mathbf{2}_n$ may be improved for each part $\tilde{V}_{(k)}$

which receives extra contracting factors $L^{-n(k/2-1)d}$ due to Cauchy estimates [see Eqs. (4.28)–(4.31)]. This procedure (the separation of $\tilde{V}$ into several parts) has already been successfully used in another problem.[7] Thus, an extra factor appears in $\mathscr{C}_n$ of (5.6) (together with δ^{n_0+n}), increasing the falloff.

As a final comment, we emphasize once more the advantage of using a representation showing the property of orthogonality between scales to study correlation functions: the simplicity of the final formulas, specifically, the structure of the dominant part and the easy analysis of the subdominant one must be mentioned.

Although we consider in this paper only scalar lattice models and the block renormalization group, we would like to see the extension of the representation obtained here to more complicated vector and fermionic models, as well as for other renormalization groups (e.g., for those considered in refs. 8 and 9).

It should also be mentioned that in ref. 10 wavelets are constructed and related to the Gaussian fixed point.

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